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ON THE MOTION OF AN ELLIPSOID ON A ROUGH SURFACE WITH SLIPPAGE*

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Method of averaging is used to investigate the motion with slippage of a homogeneous, triaxial and almost spherical ellipsoid on a fixed horizontal plane in the presence of a small amount of dry friction. The first approximation equations are very complicated and their full integration is not performed. Two first integrals of the averaged equations are found, general geometrical properties of the motion investigated and simplest particular solutions of the averaged equations considered. The qualitative and quantitative investigation of the tendency of the ellipsoid to rotate about its largest, vertically positioned axis, is carried out. Results of the analysis of the motion of the ellipsoid in the presence of a small amount of dry friction are also formulated.

A particular case of the motion of a heavy homogeneous triaxial ellipsoid on a fixed horizontal plane was studied in /1/, where it was assumed that slippage was absent, and the point of contact between the ellipsoid and the plane described on the surface of the ellipsoid one of its principal cross sections. In /2,3/ the author investigated a motion of an almost spherical ellipsoid. Periodic motions without slippage are studied, generated by the stationary motions of a homogeneous sphere and an averaging method is used to study a general case of the motion. In the case of a perfectly smooth plane, methods of Hamiltonian mechanics are used to establish the character of the motion of the ellipsoid over an infinitely long period of time.

1. Let us write the equations necessary for solution of the problem of motion with slippage, of an arbitrary, convex, heavy solid on a fixed rough horizontal plane. Let $OXYZ$ be a fixed coordinate system with the origin at some point on the plane, and a vertical OZ axis. We denote the unit vector of the vertical by $\mathbf{n}$ which is a unit vector of the outward normal to the surface of the body, constructed at the point P of contact between the body and the plane. We attach to the solid the $Cxyz$ coordinate system with the origin at its center of gravity C and the axes directed along the principal central axes of inertia of the body. The orientation of the body relative to the fixed coordinate system is determined with help of the matrix $\mathbf{A}$ of direction cosines

$$\begin{Vmatrix} X \\ Y \\ Z \end{Vmatrix} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} \tag{1.1}$$

In the $Cxyz$ coordinate system the vector $\mathbf{CP}$ has components x, y, and z. We assume that the equation of the surface surrounding the body has the form

$$\varphi(x, y, z) = 0 \tag{1.2}$$

Then

$$\mathbf{n} = \operatorname{grad}\varphi / |\operatorname{grad}\varphi|, \quad \mathbf{n}' = -(a_{31}, a_{32}, a_{33}) \tag{1.3}$$

where a prime denotes transposition. Let $\mathbf{v}$ and $\mathbf{v}_c$ denote the velocity vectors of the points P and C of the body, and v_X, v_Y, v_Z and $X_c^{\boldsymbol{\cdot}}, Y_c^{\boldsymbol{\cdot}}, Z_c^{\boldsymbol{\cdot}}$ their components in the fixed coordinate system. Then

$$\mathbf{v} = \mathbf{v}_c + \mathbf{A}\boldsymbol{\omega} \times \mathbf{CP} \tag{1.4}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity vector of the body, given in the $Cxyz$ coordinate system by its components p, q, and r. Let R_X, R_Y, R_Z be the components of the reaction $\mathbf{R}$ of the plane in the $OXYZ$ coordinate system. Then we have, in the case of dry friction,

$$R_X = -fR_Z\cos\theta, \quad R_Y = -fR_Z\sin\theta; \quad v_X = v\cos\theta, \quad v_Y = v\sin\theta \tag{1.5}$$

where f is the coefficient of friction and is a constant, and θ denote the angle between the

*Prikl.Matem.Mekhan.,Vol.47,No.2,pp.310-320,1983 velocity vector $\mathbf{v}$ of the point of contact and the OX axis of the fixed coordinate system. We assume that $v \neq 0$, i.e., the motion is carried out with slippage. The coupling equation expressing the fact that the component v_Z of velocity at the point of contact is zero, can be written with help of equations (1.1), (1.3) and (1.4) in the form of the following kinematic relation:

$$Z_c^{\cdot} + a_{31}(qz - ry) + a_{32}(rx - pz) + a_{33}(py - qx) = 0 \tag{1.6}$$

The theorems on the change of angular momentum and kinematic momentum yield two vector equations

$$m\mathbf{v}_c^{\cdot} = mg\mathbf{n} + \mathbf{R}, \quad \mathbf{G}^{\cdot} + \boldsymbol{\omega} \times \mathbf{G} = \mathbf{M}; \quad \mathbf{M} = \mathbf{CP} \times \mathbf{R} \tag{1.7}$$

where m denotes the mass of the body, g is acceleration of free fall and $\mathbf{G}$ is kinetic moment of the body relative to the center of gravity. In the $Cxyz$ system we have $\mathbf{G}' = (Ap, Bq, Cr)$ where A, B, and C are the principal central moments of inertia of the body. We denote by $\mathbf{M}$ in (1.7) the moment of reaction of the plane relative to the center of gravity, and write (1.7) in the scalar form in the following equations

$$mX_c^{\cdot\cdot} = -fR_Z\cos\theta, \; mY_c^{\cdot\cdot} = -fR_Z\sin\theta, \; mZ_c^{\cdot\cdot} = R_Z - mg \tag{1.8}$$

$$Ap^{\cdot} + (C - B)\,qr = M_x \;\; \{pqr,\, xyz,\, ABC\} \tag{1.9}$$

$$M_x = [(a_{33}y - a_{32}z) + f(a_{22}z - a_{23}y)\sin\theta + f(a_{12}z - a_{13}y)\cos\theta]\,R_Z \tag{1.10}$$

$$\{xyz,\, a_{i1}a_{i2}a_{i3}\;(i = 1, 2, 3)\}$$

Another two equations not appearing in (1.9) and (1.10) but are obtained from them by simultaneous cyclic permutation of the indices shown within the curly brackets.

Let us also write the kinematic Poisson equations

$$a_{i1}^{\cdot} = a_{i2}r - a_{i3}q, \; a_{i2}^{\cdot} = a_{i3}p - a_{i1}r, \; a_{i3}^{\cdot} = a_{i1}q - a_{i2}p \tag{1.11}$$
$$(i = 1, 2, 3)$$

The equations (1.2)–(1.6), (1.8)–(1.11) form a closed system of equations describing the problem of motion with slippage of an arbitrary convex, heavy solid on a fixed rough surface, in the presence of dry friction.

2. Let a moving body represent a homogeneous ellipsoid the surface of which is given in the $Cxyz$ system by the equation

$$\varphi \equiv x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 \tag{2.1}$$

In the case of $a = b = c$ we obtain the well studied /4–6/ problem of motion with friction of a homogeneous sphere on a horizontal plane (billiard ball). If at the initial instant the instantaneous angular velocity vector $\boldsymbol{\omega}$ is perpendicular to the velocity vector $\mathbf{v}_c$ of the center of the sphere, then the latter moves along a straight line, otherwise it moves along a parabola. The slippage ceases at the instant $t = 2v_0/(7fg)$, where v_0 is the initial velocity of the point of contact. Beginning from this moment, the motion will consist of rolling with spinning.

Let the ellipsoid differ little from a sphere of radius l, and let the friction coefficient f be small. We use the quantity max $\{|a - b|/l, |b - c|/l, |c - a|/l, f\}$ as the small parameter ε. When $\varepsilon = 0$, we have the problem of motion of a sphere on a smooth horizontal plane; the sphere center moves uniformly and rectilinearly and the sphere rotates uniformly about a fixed direction in the fixed coordinate system. Assuming that the above motion is a generating motion, we study the motion of the ellipsoid at $0 < \varepsilon \ll 1$ by asymptotic methods. To do this, we transform the equations of Sect.1 to the form suitable for the application of the averaging method /7/.

From (2.1), (1.3) and (1.6) it follows that $Z_c^{\cdot\cdot}$ is a quantity of first order of smallness in ε, therefore, according to the third equation of (1.8), the normal reaction of the plane R_Z is equal to the weight of the ellipsoid with an error of the order of ε Neglecting terms of the order of ε^2, we can write the first two equations of (1.8) in the form

$$X_c^{\cdot\cdot} = -fg\cos\theta, \; Y_c^{\cdot\cdot} = -fg\sin\theta \tag{2.2}$$

Performing the variable change

$$x = ax', \; y = by', \; z = cz' \tag{2.3}$$

and remembering that for a homogeneous ellipsoid

$$A = m(b^2 + c^2)/5, \; B = m(c^2 + a^2)/5, \; C = m(a^2 + b^2)/5 \tag{2.4}$$

we find from (2.1), (1.3), (1.10) that to write the equations (1.9) with the accuracy of up to the terms of order ε it is sufficient to put in their right hand parts

$$M_x = 5g\,(B - C)\,y'z'/l + fmgl\,(a_{21}\cos\theta - a_{11}\sin\theta) \qquad (2.5)$$
$$\{xyz,\ x'y'z',\ ABC,\ a_{i1}a_{i2}a_{i3}\ (i = 1, 2)\}$$

The equation of the surface of the ellipsoid (2.1) will become, in the variables x', y', z', an equation of a sphere

$$x'^2 + y'^2 + z'^2 = 1 \qquad (2.6)$$

From (1.11) we obtain (with $i = 3$) with help of (1.3) and (2.3), the following equations for the variables x', y', z' in the first approximation in ε:

$$x'^{\cdot} = y'r - z'q + g_1,\ y'^{\cdot} = z'p - x'r + g_2,\ z'^{\cdot} = x'q - y'p + g_3 \qquad (2.7)$$
$$g_1 = 2\,(c - b)\,x'y'z'p/l + (a - c)\,(2x'^2 - 1)\,z'q/l + (b - a)\,(2x'^2 - 1)\,y'r/l$$
$$\{g_1g_2g_3,\quad abc,\ x'y'z',\ pqr\}$$

and the equations are dependent by virtue of (2.6). From relations $v_X = v\cos\theta,\ v_Y = v\sin\theta$ we have $\theta^{\cdot} = (v_Y^{\cdot}\cos\theta - v_X^{\cdot}\sin\theta)/v,\ v^{\cdot} = v_Y^{\cdot}\sin\theta + v_X^{\cdot}\cos\theta$. Substituting here the derivatives $v_X^{\cdot}, v_Y^{\cdot}$ obtained by differentiating the kinematic relation (1.4) and using the equations (1.11), (2.2), (2.3), (2.7) as well as the equations connecting the firection cosines a_{ij}, we obtain the following differential equations for θ and v in the first approximation in ε:

$$\theta^{\cdot} = (\Phi\cos\theta + \Psi\sin\theta)/v,\ v^{\cdot} = -{}^7/_2fg + (\Phi\sin\theta - \Psi\cos\theta) \qquad (2.8)$$
$$\Phi = \varphi_1 + \varphi_2 + \varphi_3,\ \Psi = \psi_1 + \psi_2 + \psi_3$$
$$\varphi_1 = (a - c)\,[a_{12}\,(pr + 5gx'z'/l) - 2x'z'q\,(a_{11}p + a_{12}q + a_{13}r) + 2a_{22}q\,(x'p - z'r) - 2q^2\,(a_{21}x' - a_{23}\,z')]$$
$$\psi_1 = (a - c)\,[a_{22}\,(pr + 5gx'z'/l) + 2x'z'q\,(a_{21}p + a_{22}q + a_{23}r) - 2a_{12}q\,(x'p - z'r) + 2q^2\,(a_{11}x' - a_{13}z')]$$
$$\{\varphi_1\varphi_2\varphi_3,\ \psi_1\psi_2\psi_3,\ abc,\ pqr,\ x'y'z',\ a_{i1}a_{i2}a_{i3}\ (i = 1, 2)\}$$

Following /2,3/ we replace the variables x', y', z' by ρ, ζ, γ using the following formulas:

$$\begin{Vmatrix} x' \\ y' \\ z' \end{Vmatrix} = \begin{Vmatrix} \sin\beta & \cos\alpha\cos\beta & \sin\alpha\cos\beta \\ -\cos\beta & \cos\alpha\sin\beta & \sin\alpha\sin\beta \\ 0 & -\sin\alpha & \cos\alpha \end{Vmatrix} \begin{Vmatrix} \rho\sin\gamma \\ \rho\cos\gamma \\ \zeta \end{Vmatrix} \qquad (2.9)$$
$$\sin\alpha = \sqrt{p^2 + q^2}/\omega,\ \cos\alpha = r/\omega,\ \sin\beta = q/\sqrt{p^2 + q^2},$$
$$\cos\beta = p/\sqrt{p^2 + q^2}$$

The variables ρ and ζ are connected with each other by the relation

$$\rho^2 + \zeta^2 = 1 \qquad (2.10)$$

When $\varepsilon = 0\ \gamma^{\cdot} = \omega$ and the quantities ρ and ζ are constant, with ρ denoting the distance from the center of the ellipsoid (sphere) along the straight line passing through the point of contact in the direction parallel to the vector $\boldsymbol{\omega}$, then $|\zeta|$ describes the distance from the center of the ellipsoid (sphere) along a plane perpendicular to $\boldsymbol{\omega}$ and passing through the point of contact.

We carry out a variables change in (1.11) using (2.9) in which we replace x', y', z' by $a_{i1}, a_{i2}, a_{i3}\ (i = 1, 2, 3)$ respectively, and ρ, ζ, γ by ρ_i, ξ_i, γ_i. The quantities $\zeta_1, \zeta_2, \zeta_3$ are cosines of the angles between the vector $\boldsymbol{\omega}$ and the axes OX, OY, OZ of the fixed coordinate system. We have the following identities:

$$\rho_i^2 + \zeta_i^2 = 1 \qquad (2.11)$$

In the first approximation in ε the variables $\zeta, \zeta_i\ (i = 1, 2, 3)$ satisfy the following differential equations:

$$\zeta^{\cdot} = (x'p^{\cdot} + y'q^{\cdot} + z'r^{\cdot})/\omega - (pp^{\cdot} + qq^{\cdot} + rr^{\cdot})/\omega^2 + (pg_1 + qg_2 + rg_3)/\omega \qquad (2.12)$$
$$\zeta_i^{\cdot} = (a_{i1}p^{\cdot} + a_{i2}q^{\cdot} + a_{i3}r^{\cdot})/\omega - (pp^{\cdot} + qq^{\cdot} + rr^{\cdot})/\omega^2 \qquad (2.13)$$

The quantities $p^{\cdot}, q^{\cdot}, r^{\cdot}$ appearing here must be obtained from equations (1.9), with the right hand sides given by (2.5). Neglecting the terms of order ε and higher, we also obtain

$$\gamma_i^{\cdot} = \gamma^{\cdot} = \omega \qquad (2.14)$$

Let us replace ζ_1, ζ_2 by variables α_1, α_2 according to the formulas

$$\alpha_1 = \zeta_1 \cos\theta + \zeta_2 \sin\theta, \quad \alpha_2 = -\zeta_1 \sin\theta + \zeta_2 \cos\theta \tag{2.15}$$

The quantity α_1 is a cosine of the angle between the vectors $\boldsymbol{\omega}$ and $\mathbf{v}$, while α_2 is a cosine between $\boldsymbol{\omega}$ and the vector perpendicular to $\mathbf{v}$ and lying in the horizontal plane, with the smaller angle of rotation from $\mathbf{v}$ to this vector counted in the anticlockwise direction. From (2.15) and (2.13) we obtain for α_1, α_2 the following equations:

$$\begin{aligned} \alpha_1^{\cdot} &= [(a_{21}p^{\cdot} + a_{22}q^{\cdot} + a_{23}r^{\cdot}) \sin\theta + (a_{11}p^{\cdot} + a_{12}q^{\cdot} + \\ &\quad a_{13}r^{\cdot}) \cos\theta]/\omega - (pp^{\cdot} + qq^{\cdot} + rr^{\cdot})\alpha_1/\omega^2 + \theta^{\cdot}\alpha_2 \\ \alpha_2^{\cdot} &= [(a_{21}p^{\cdot} + a_{22}q^{\cdot} + a_{23}r^{\cdot}) \cos\theta - (a_{11}p^{\cdot} + a_{12} q^{\cdot} + \\ &\quad a_{13}r^{\cdot}) \sin\theta]/\omega - (pp^{\cdot} + qq^{\cdot} + rr^{\cdot})\alpha_2/\omega^2 - \theta^{\cdot}\alpha_1 \end{aligned} \tag{2.16}$$

Equations (1.9) (with the right-hand sides (2.5)), (2.2), (2.8), (2.12)—(2.14) and (2.16) represent a system written in a form suitable for use with the method of averaging. In the equations (2.12), (2.13) (2.16) $p^{\cdot}, q^{\cdot}, r^{\cdot}$ are function obtained from (1.9), x', y', z', a_{ij} are assumed to be given in terms of $\rho, \zeta, \gamma, \rho_i, \zeta_i, \gamma_i, p, q, r$ according to the formulas (2.9) defining the variable change, and $\theta^{\cdot}$ in (2.16) represents the right-hand side of the first equation of (2.8). The variables $X_c^{\cdot}, Y_c^{\cdot}, p, q, r, \zeta, \zeta_i, \theta, v$ in the resulting system of equations are slow, while γ, γ_i are fast. Let us average the right-hand sides of the equations for the slow variables, over the fast variables, Taking also into account (2.10) and (2.11) and the relations $\rho\rho_i \cos(\gamma - \gamma_i) + \zeta\zeta_i = 0 \ (i = 1, 2)$ valid for the unperturbed (at $\varepsilon = 0$) motion, we obtain the following averages system of equations:

$$X_c^{\cdot\cdot} = -fg\cos\theta, \ Y_c^{\cdot\cdot} = -fg\sin\theta \tag{2.17}$$

$$Ap^{\cdot} + [1 + 5g\,(3\zeta^2 - 1)/(2\omega^2 l)]\,(C - B)\,qr = f\,mgl\,\alpha_2 p/\omega \tag{2.18}$$

$$\{ABC, pqr\}$$

$$\alpha_1^{\cdot} = -\frac{5fg}{2\omega l}\alpha_1\alpha_2 + \frac{5g\zeta F}{2l\omega^2}\left(\frac{\alpha_2}{v} + \frac{1}{\omega l}\right)\alpha_2 \tag{2.19}$$

$$\alpha_2^{\cdot} = \frac{5fg}{2\omega l}(1 - \alpha_2^2) - \frac{5g\zeta F}{2l\omega^2}\left(\frac{\alpha_2}{v} + \frac{1}{\omega l}\right)\alpha_1$$

$$\theta^{\cdot} = \frac{5g\zeta F}{2vl\omega^2}\alpha_2, \quad v^{\cdot} = -\frac{7}{2}fg + \frac{5g\zeta F}{2l\omega^2}\alpha_1 \tag{2.20}$$

$$\zeta^{\cdot} = -\frac{5fg\zeta}{2l\omega}\alpha_2, \quad \zeta_3^{\cdot} = -\frac{5fg\zeta_3}{2l\omega}\alpha_2$$

The averaged equations for ζ_1 and ζ_2 are not written out, since they are of no further use to us. From (2.18) we obtain the following auxilliary equations for ω:

$$\omega^{\cdot} = \frac{5fg}{2l}\alpha_2 \tag{2.21}$$

and in (2.19), (2.20) we have used the notation

$$F = (a - c)(p^2 - r^2) + (b - a)(q^2 - p^2) + (c - b)(r^2 - q^2) \tag{2.22}$$

Solutions of the averaged system approximate the slow variables with an error of the order of ε over the time interval of the order of ε^{-1}. It can be shown that the averaged system has the following integrals:

$$\zeta\omega = \text{const}, \ \zeta_3\omega = \text{const} \tag{2.23}$$

The second of these integrals means that the projection of the instantaneous velocity vector on the vertical is constant in the first approximation. Since the general solution of the averaged system is hardly possible, we shall limit ourselves to finding its particular solutions and establishing certain general properties of the motion of the ellipsoid.

3. First we establish some geometrical properties of the motion. Let us carry out in (2.18) the variable change according to the formulas

$$p = \omega\alpha_x, \ q = \omega\alpha_y, \ r = \omega\alpha_z \tag{3.1}$$

where $\alpha_x, \alpha_y, \alpha_z$ are the cosines of the angles between the vector $\boldsymbol{\omega}$ and the axes Cx, Cy, Cz of the ellipsoid respectively. From (2.18) and (2.21) we obtain

$$A\alpha_x + \omega\,[1 + 5g\,(3\zeta^2 - 1)/(2\omega^2 l)]\,(C - B)\,\alpha_y\alpha_z = 0 \qquad (3.2)$$
$$\{ABC,\ xyz\}$$

It follows that in the first approximation the quantities $\alpha_x, \alpha_y, \alpha_z$ describing the orientation of the vector $\boldsymbol{\omega}$ relative to the ellipsoid can be found using the formulas used in determining the quantities p, q, r in the Euler-Poinsot motion in which the part of time is played by

$$\tau = \int_0^t \omega\left[1 + 5g\,\frac{3\zeta^2 - 1}{2\omega^2 l}\right] dt \qquad (3.3)$$

Let the components of the vector $\mathbf{G}$ in the fixed coordinate system be G_X, G_Y, G_Z. The theorem on the change of kinetic momentum yields for these components the following equations:

$$G_X^{\cdot} = a_{11}M_x + a_{12}M_y + a_{13}M_z \ \{XYZ, a_{1i}, a_{2i}a_{3i}\ (i = 1, 2, 3)\} \qquad (3.4)$$

All components G_X, G_Y, G_Z are slow variables. Using (3.4) we obtain a differential equation for G, and averaging its right-hand side yields the following first approximation equation:

$$G^{\cdot} = fmgl\alpha_2 \qquad (3.5)$$

Averaging the right-hand side of the expression for the derivative of the kinetic energy of motion relative to the center of gravity, yields the following first approximation equation:

$$T^{\cdot} = fmgl\omega\alpha_2 \qquad (3.6)$$

Connecting mentally the ellipsoid in question with its inertia ellipsoid relative to the center of gravity, passing through the center of gravity a straight line parallel to the vector $\boldsymbol{\omega}$ and projecting through the point of intersection of this line with the inertia ellipsoid a plane tangent to this ellipsoid we find, just as in the Euler-Poinsot case, that the plane will be perpendicular to the vector $\mathbf{G}$ and will lie at the distance $d = \sqrt{2T}/G^2$ from the center of the ellipsoid. In the Euler-Poinsot case T and G are constant, hence so is d, while in the present case T and G both vary with time. However, the calculations employing (3.5) and (3.6) and the closeness of the moments of inertia (2.4) shown that $d^{\cdot} = 0$ with the accuracy of up to the terms of order ε inclusive. Therefore we find that in the present case, just as in the Euler-Poinsot case, the inertia ellipsoid rolls and rotates without slippage over the tangent plane constructed, the latter remaining at a distance from the center of the ellipsoid, unchanged in the first approximation. In the present case however, the center of the ellipsoid moves in accordance with the equations (2.17) and the orientation of the vector $\mathbf{G}$ varies relative to the fixed $OXYZ$ coordinate system.

Let us obtain the equations determining the orientation of the vector $\mathbf{G}$. Computing the right-hand side of the third equation of (3.4) using (1.3), (2.1), (2.3) and (2.5) shows that it is equal to zero with the accuracy of up to the terms of order ε inclusive. This implies that the projection G_Z of the kinetic moment vector on the vertical is constant in the first approximation. Let σ be the angle between the OX axis and the projection of $\mathbf{G}$ on the horizontal plane. We see that

$$\sigma^{\cdot} = \frac{G_X G_Y^{\cdot} - G_Y G_X^{\cdot}}{G^2 - G_Z^2} \qquad (3.7)$$

Replacing here $G_X^{\cdot}$ and $G_Y^{\cdot}$ by the right-hand sides of the corresponding equations of (3.4) and averaging, we obtain the following first approximation equation:

$$\sigma^{\cdot} = \frac{2m^2 l^2 g}{5\omega\,(G^2 - G_Z^2)}\,[f\omega^2 l\alpha_1 - \zeta\,(1 - \zeta^2)\,F] \qquad (3.8)$$

4. We know that a rapidly spinning symmetrical top placed on a rough horizontal plane develops a tendency to raise its center of gravity and tends to rotate about its vertical symmetry axis. It appears that the first correct explanation of the phenomenon was given in /8/ where the discussion was carried out from the almost modern positions /9-11/. Chapter 18 of /12/ deals with the problems of mathematical theory explaining the rise of the axis of a symmetric top. The axis rise effect appears also in the case when the top is not necessarily symmetrical. The author of /13/ describes an experiment carried out by Thomson with an ellipsoidal stone. The motion of the stone spun rapidly on a rough horizontal plane evolved to that the stone exhibited a tendency to rotate about its longest axis which, in turn, strove

to occupy a vertical position, provided that the rotation was sufficiently rapid.

Let us now consider the tendency of the ellipsoid to rotate about its longest vertically situated axis. Using first the integrals (2.23) of the averaged equations of motion, we arrive at certain qualitative conclusions concerning the evolution of the motion of the ellipsoid, not necessarily rapidly spun, but for the most general case of its motion. If its angular velocity ω decreases, then the constancy of the value of the projection of the vector $\boldsymbol{\omega}$ on the vertical implies that the ellipsoid tends to assume a vertical position. Further, from the first integral of (2.23) it follows that when ω decreases, then the quantity $|\zeta|$ must increase. Taking into account the geometrical meaning of the variable ζ, we conclude that when ω decreases, then the ellipsoid tends to rise and stand on its longest, vertically oriented axis. When ω increases, we have the opposite phenomenon, the vector $\boldsymbol{\omega}$ and the longest axis of the ellipsoid both show a tendency to increase their deviation from the vertical.

Analysis of the consequences of the existence of the integrals (2.23) is insufficient to obtain quantitative results regarding the evolution of the motion of the ellipsoid, and the averaged equations themselves must be used. Let $|v_c| \ll |\boldsymbol{\omega} \times \mathbf{CP}|$ when $t = 0$. Then the quantity α_2 is negative and α_1 small at the initial instant. We shall assume that $|\alpha_1|$ is not less than the first order in ε. Then from (2.19) we obtain the following first approximation equation:

$$\alpha_2^{\cdot} = 5fg(1 - \alpha_2^2)/2\omega l \tag{4.1}$$

which forms, together with (2.21), a closed system of equations. Denoting by zero subscript the initial values of the variables, we obtain the general solution of this system

$$\omega = (\omega_0^2 + 2\alpha_{20}\omega_0\tau + \tau^2)^{1/2}, \quad \alpha_2 = \frac{\alpha_{20}\omega_0 + \tau}{\omega}\left(\tau = \frac{5fg}{2l}t\right) \tag{4.2}$$

The quantity ω decreases with increasing t, and α_2 increases from its initial negative value α_{20}. It remains negative up to the time $t_1 = 2\omega_0 l\,|\alpha_{20}|/(5fg)$. Geometrical considerations imply that $\alpha_{20} = -\rho_0$, $v_0 = \rho_0 l\omega_0$ with an error of the order of ε, therefore we have approximately $t_1 = 2v_0/(5fg)$. When α_1 are small, then from the second equation of (2.20) we find that in the first approximation, just as in the case of a sphere on a rough plane, the velocity of the point of contact is

$$v = v_0 - {}^7/_2 fgt \tag{4.3}$$

The velocity v becomes zero at the instant $t_2 = 2v_0/(7fg)$, and a motion without slippage commences. Since $t_1 > t_2$, it follows that the angular velocity ω decreases over the whole time interval $0 < t < t_2$ and the ellipsoid continues to rise onto its longest axis up to the onset of the motion without slippage. Let us estimate the amount Δt of time necessary for the ellipsoid to turnover from its shortest semiaxis a to its longest semiaxis c. This means that $|\zeta|$ should change over the period Δt by the amount equal to $(c - a)/l$. From (2.23) we have $\zeta(t) = \zeta_0\omega_0/\omega(t)$. Setting here $\zeta_0 = a/l$, $\zeta(\Delta t) = c/l$ we obtain, neglecting the terms of order ε^2 and higher, $\omega(\Delta t) = \omega_0[1 - (c-a)/l]$. Substituting this value ω into the left-hand side of the first equation of (4.2), we obtain

$$\Delta t = \frac{2(c-a)}{5fg\,|\alpha_{20}|}\omega_0 \tag{4.4}$$

Terms neglected from the right-hand side of (4.4) are smaller than the retained terms by at least one order of magnitude ε. In order to have the turnover continuing up to the onset of the motion without slippage, it is necessary to demand that the inequality $\Delta t < t_2$ holds and this, with the approximate relations $\alpha_{20} = -\rho_0$, $v_0 = \rho_0\omega_0 l$ taken into account, implies that the following inequality must hold:

$$\rho_0 > \sqrt{7(c-a)/(5l)} \tag{4.5}$$

Thus, in order to make possible the turnover of the ellipsoid from the shortest to the longest semiaxis, we must require that the angle between the smallest semiaxis and the vector $\boldsymbol{\omega}$ be not too acute at the initial instant, otherwise the time Δt necessary for the turnover of the ellipsoid will exceed the time t_2 at which motion without slippage begins. This conclusion can be reached in a qualitative manner directly from (2.21) and (2.23): the smaller ρ_0, the smaller $|\alpha_{20}|$ and, according to (2.21), the slower the decrease in the value of ω and hence, according to (2.23), the slower the increase in $|\zeta|$.

5. Let us consider the case of perfectly smooth plane. From (1.8) with $f = 0$ we obtain $X_c^{\cdot} = \text{const}$, $Y_c^{\cdot} = \text{const}$. Third equation of (2.20) and (2.21) together imply that in the first approximation with $f = 0$, do not decrease the accuracy, we can write $\zeta = \zeta_0$, $\omega = \omega_0$ in the square brackets of (2.18). Already in the first approximation the ellipsoid executes an Euler–Poinsot motion about the vector $\mathbf{G}$, in which the following quantity is regarded as time:

$$\tau = [1 + 5g(3\zeta_0^2 - 1) / (2\omega_0^2 l)]\, t \tag{5.1}$$

and depends on the initial condititions.

From the constancy of G_Z and the fact that according to (3.5) the quantity G is constant at $f = 0$ also in the first approximation, it follows that the angle between the vector G and the vertical is also constant in the first approximation. The function (2.22) can be written with the accuracy of the order of ε^2, in the form

$$F = 5\,[(A + B + C)\,\omega^2 - 6T]/(2ml)$$

When $f = 0$ we find, from (3.6), that we can, without affecting the accuracy, write in the right-hand side of (3.8) not only $\omega = \omega_0 = \mathrm{const}$, but also $F = F_0 = \mathrm{const}$. Remembering also that the projection of G on the horizontal plane is equal to $^2/_5 ml^2\rho_0\omega_0$, with the accuracy of the order of ε, we find from (3.8) that when $f = 0$, the quantity $\sigma^{\cdot}$ is constant in the first approximation and given by the formula

$$\sigma^{\cdot} = -5g\zeta_0 F_0/(2l^2\omega_0^3) \tag{5.2}$$

Thus we find that when $f = 0$, the projection of the center of gravity of the ellipsoid on the horizontal plane is uniform and rectilinear, and the ellipsoid itself moves about the kinetic moment vector in accordance with Euler–Poinsot with an altered time scale (5.1), while the kinematic moment vector is constant in modulo and precesses slowly about the vertical at a constant angular velocity (5.2), remaining at a constant angular distance from it. The same result was obtained by a different method in /2/.

6. We shall indicate some simplest particular solutions of the averaged system (2.17)–(2.20). First we consider a solution in which $\zeta = 0$. This corresponds to a motion of the ellipsoid in which we can assume, with an error of the order of ε, that its center of gravity lies in the plane perpendicular to ω and passing through the point of contact between the ellipsoid and the plane. When $\zeta = 0$, the third equation of (2.20) is satisfied identically, while (2.17) and the first two equations of (2.20) together imply that, as in the case when a sphere moves with slippage on a rough plane, $\theta = \theta_0 = \mathrm{const}$, $v = v_0 - {}^7/_2 fgt$ and the projection of the center of gravity on the plane moves either along a straight line, or along a parabola, depending on the intial conditions. When $\zeta = 0$, the equations (2.18) and (2.19) can be written as

$$Ap^{\cdot} + (1 - 5g/(2\omega^2 l))\,(C - B)\,qr = f\, mgl\alpha_2 p/\omega \quad \{ABC,\ pqr\} \tag{6.1}$$

$$\alpha_1^{\cdot} = -5fg\alpha_1\alpha_2/(2\omega l), \quad \alpha_2^{\cdot} = 5fg\,(1 - \alpha_2^2)/(2\omega l) \tag{6.2}$$

From (2.21) and (6.2) we find that ω and α_2 vary with time according to (4.2), while $\alpha_1 = \alpha_{10}\omega_0/\omega$. The orientation of the instantaneous angular velocity vector is given in the $OXYZ$ coordinate system by equations

$$\zeta_1 = \alpha_1 \cos\theta_0 - \alpha_2 \sin\theta_0, \quad \zeta_2 = \alpha_1 \sin\theta_0 + \alpha_2 \cos\theta_0, \quad \zeta_3 = \zeta_{30}\omega_0/\omega \tag{6.3}$$

The quantities p, q, and r (with the relation $\omega(t)$ known) are obtained from (3.1) and equations (3.2), which can be reduced with help of the independent variable

$$\tau = \int_0^t \omega\,(1 - 5g/(2\omega^2 l))\, dt$$

to a system in the Euler–Poinsot problem, integrable in terms of elliptical functions.

7. It can be confirmed that the averaged system of equations admits a solution for which $p = 0$, $q = 0$ and the quantities $X_c^{\cdot}, X_c^{\cdot}, \alpha_1, \alpha_2, \theta, V, \zeta, \zeta_3$, $\omega = |r|$ satisfy the system of equations (2.17), (2.19)–(2.21) in which the quantity F/ω^2 is replaced by the constant $2c - a - b$. This particular solution corresponds to a motion in which the vector ω remains parallel to one of the axes of the ellipsoid throughout the motion. If the semiaxes of the ellipsoid are connected by the relation $a + b = 2c$, then $\theta = \theta_0 = \mathrm{const}$, the projection of the center of gravity on the plane moves along a straight line or a parabola, and the quantities $\alpha_1, \alpha_2, v, \omega$ change with time just as in Sect.6 where $\zeta = 0$.

8. We shall indicate another interesting particular solution of the averaged system corresponding to the motion of an ellipsoid with constant instantaneous angular velocity vector ω. We obtain the solution, and conditions of its existence. From (2.18)–(2.21) we find that the solution in question has the following analytic expression:

$$p = p_0,\ q = q_0,\ r = r_0,\ \alpha_1 = fl\omega_0^2/(\zeta_0 F_0),\ \alpha_2 = 0 \tag{8.1}$$
$$\theta = \theta_0,\ \zeta = \zeta_0,\ v = v_0 - fgt$$

The projection of the center of gravity on the plane moves along a straight line or a parabola. The constants ζ_0, ω_0 in (8.1) are connected by a relation which causes the expression within the square brackets in (2.18) to vanish

$$\zeta_0^2 = 1/_3 - 2\omega_0^2 l/(15g) \tag{8.2}$$

Requiring now that the right-hand side of (8.2) be positive and taking into account the fact that $|\alpha_1| \leqslant 1$, we obtain the upper bound for the angular velocity ω_0 and the coefficient of friction f

$$\omega_0 < \sqrt{\frac{5g}{2l}}, \quad f \leqslant \frac{|\zeta_0|}{l} \left|(a+b+c) - 3\frac{ap_0^2 + bq_0^2 + cr_0^2}{\omega_0^2}\right| \tag{8.3}$$

The inequalities (8.3) represent the conditions of existence of the motion with constant vector $\boldsymbol{\omega}$. We note that in the case of a sphere with $f \neq 0$, there exists no motion with slippage at constant $\boldsymbol{\omega}$.

9. A motion of the ellipsoid was also studied under the assumption that the ellipsoid is again almost spherical and the friction is low, though not dry but viscous. We shall formulate briefly the fundamental results of investigation.

The reaction of the plane will now be given in the $OXYZ$ coordinate system by the components $-kmv\cos\theta, -kmv\sin\theta, R_Z$ where $k > 0$ is a constant and small (of the order of ε) coefficient of friction. The averaged equations of motion are obtained from the equations (2.17)–(2.21.), provided that we replace in their right-hand sides terms containing the factor fg, by the same terms but containing the factor kv. The geometrical characteristics of the motion discussed in Sect.3 also apply in the case of viscous friction, but the coefficient fg in the right-hand sides of (3.5) and (3.8) must be replaced by kv. The integrals (2.23) hold for the averaged equations just as in the case of dry friction, and as before there is a tendency of the ellipsoid to rotate about its largest, vertically positioned axis. Only, in the case of viscous friction the quantity τ in (4.2) must be determined by the equation

$$\tau = \frac{5v_0}{7l}\left(1 - e^{-7kt/2}\right) \tag{9.1}$$

while the formula (4.3) describing the decrease in the velocity of the point of contact at small $|\alpha_1|$, becomes

$$v = v_0 e^{-7kt/2} \tag{9.2}$$

Since v does not vanish at any t, it follows that we cannot have a motion without slippage, The estimation of "time" $\Delta\tau$, necessary for the turnover of the ellipsoid from the smallest to the largest axis will, in the case of viscous friction, be

$$\Delta\tau = (c - a)\,\omega_0/(\rho_0 l) \tag{9.3}$$

The above quantity must not exceed the largest possible value of τ equal to $5v_0/(7l)$, and from this, just as during the dry friction, follows the condition (4.5). The averaged equation admists the particular solution discussed in Sect.6 in which $\zeta = 0$. The variables $\alpha_1, \alpha_2, \omega$ in this solution are obtained from the same formulas as in the case of dry friction, but with τ given by equality (9.1). The velocity of the point of contact is given by (9.2), the angle $\theta = \theta_0 = \text{const}$ and the trajectory of the projection of the center of gravity on the plane are given by

$$\begin{aligned} X_c(t) &= \frac{4}{49k} v_0\cos\theta_0\left(1 - e^{-7kt/2}\right) + \left(X_{c0}^{\cdot} - \frac{2}{7} v_0\cos\theta_0\right)t + X_{c0} \\ Y_c(t) &= \frac{4}{49k} v_0\sin\theta_0\left(1 - e^{-7kt/2}\right) + \left(Y_{c0}^{\cdot} - \frac{2}{7} v_0\sin\theta_0\right)t + Y_{c0} \end{aligned} \tag{9.4}$$

Another particular solution exists, for which the vector $\boldsymbol{\omega}$ is parallel to one of the axes of the ellipsoid. In the case of viscous friction the motion with constant vector $\boldsymbol{\omega}$ discussed in Sect.8 does not exist.

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